Research Paper

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Estimating Volatility for Long Holding Periods. [1]

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ABSTRACT Sample volatilities calculated from short-interval (daily) data do not necessarily imply much about volatility of financial assets over long (yearly) horizons. In this note we construct a model-free volatility estimator to investigate the long horizon volatility of various short-term interest rate time series and study implications for short-rate models.

1 Introduction

The problem of estimating volatility is one of the most important topics in modern finance. Accurate specification of volatility is a prerequisite for modelling financial time series, such as interest rates or stocks, and crucially affects the pricing of contingent claims. Modelling volatility has therefore be widely discussed in the financial literature, see e.g. Campell, Lo, and MacKinley [7], chapter 12, Shiryaev [21], chapter 4, or Taylor [22], chapter 3 for overviews on the subject. The main focus in these studies has been to estimate volatility over short time periods and deduce results for longer period volatility from underlying models.

In this note we address the problem of estimating volatility over longer time intervals directly. Recently several attempts have been made to address this problem, most notably work by Andersen, Bollerslev, Diebold and Labys [1, 2], who use intraday observations to estimate the distribution of daily volatility, and Drost, Nijman and Werker [12, 13], who consider temporal aggregation of GARCH processes. In contrast to these approaches we do not assume any underlying parametric model for the data-generating processes. Our only assumption is that the data-generating process is first-difference stationary. The model free approach leads to an estimator, which is insensitive to short-period contamination and only reacts to effects relevant to the time period in question. Applications of the proposed

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estimator can be found in Cochrane [9], who used the estimator to obtain a measure of the persistence of fluctuations in GNP, and Kiesel, Perraudin and Taylor [17], who estimated the long term variability of credit spreads.

Related to our estimation problem are so-called moment ratio tests, which are frequently used to investigate the (weak) efficiency of financial markets, see Campbell et al. [7], chapter 1, or Pagan [20] for surveys and Lo,MacKinlay [18] and Groenendijk et al. [16] for applications related to this investigation.

The motivation behind the estimator is as follows. From the assumption that the data-generating process x_t is first-difference stationary (i.e. contains a unit root), we obtain from Wold's decomposition (see e.g. [15], §2.10) an infinite moving average representation

$$\Delta x_t = x_t - x_{t-1} = \mu + \sum_{j=0}^{\infty} a_j \epsilon_{t-j}. \tag{1}$$

Using this representation a result by Beveridge and Nelson [3] implies that x_t can be represented as the sum of a stationary and a random walk component, i.e

$$x_t = y_t + z_t \tag{2}$$

where

$$-y_t = \left(\sum_{j=1}^{\infty} a_j\right)\epsilon_t + \left(\sum_{j=2}^{\infty} a_j\right)\epsilon_{t-1} + \left(\sum_{j=3}^{\infty} a_j\right)\epsilon_{t-2} + \ldots \tag{3}$$

$$z_t = \mu + z_{t-1} + \left(\sum_{j=0}^{\infty} a_j\right)\epsilon_t, \tag{4}$$

with (ϵ_t) a sequence of uncorrelated $(0, \sigma^2)$ random variables.

The long-period behaviour of the variance of the process x_t may differ substantially for processes with representation (2). This becomes of particular importance for valuation of contingent claims and, in case of interest rate models, for bond pricing, since the pricing formulae crucially depend on the volatility. Since, in general, the long-term behaviour of the variance of x_t is dominated by the variance of the random walk component, the use of a volatility estimator based on daily time intervals to contingent claoims/bonds longer time to maturity may lead to substantial pricing errors. In the next section we introduce the estimator and discuss some of its properties. We perform Monte Carlo experiments to illustrate the properties of the estimator in section 3. In section 4 we apply it to estimate long holding period variances for several interest rate series. By analysing the quotient of long-term to short-term variances (variance ratio) we can infer the magnitude of the random walk component in the short term interest rate process. This has implications for the appropriate modelling of the short rate and relates to recent results on the empirical verification of various short-term interest rate models, see [4, 8]. Section 5 concludes.

2 Construction and Properties of the Estimator

We start with a general representation of a first-difference stationary linear process as the sum of a stationary and a random walk component, i.e

$$x_t = y_t + z_t \tag{5}$$

with

$$\begin{aligned} y_t &= B(L)\delta_t & (6) \\ z_t &= \mu + z_{t-1} + \epsilon_t, & (7) \end{aligned}$$

with $B(L)$ a polynomial in the lag operator $L\delta_t = \delta_{t-1}$, (ϵ_t) uncorrelated, $(0, \sigma^2)$ random variables, and $\mathbb{E}(\epsilon_t \delta_t)$ arbitrary. Such a decomposition implies that $\mathbb{E}_t(x_{t+k}) \approx z_t + k\mu$. In that sense we call z_t the permanent and y_t the temporary component of x_t (compare also [7] for a related model and interpretation). This suggests that the long term variability of x_t is also dominated by the innovation variance $\sigma^2_{\Delta z}$ of the random walk component. Utilizing the Beveridge-Nelson [3] decomposition of a process x_t given by (5) one can show that the innovation variance $\sigma^2_{\Delta z}$ is invariant to the particular decomposition of type (5) chosen (in particular, only the Beveridge-Nelson decomposition is guaranteed to exist, see also [9]). To make the above arguments on the importance of the innovation variance more precise, consider the $k-$period variability. A standard argument (compare

§2.1) shows

$$\mathbb{V}ar_t(x_{t+k} - x_t) = k\gamma_0 + 2\sum_{j=1}^{k-1}(k-j)\gamma_j, \tag{8}$$

with γ_j the autocovariances of the stationary process $(\Delta x_t) = (x_t - x_{t-1})$. Then

$$\lim_{k\to\infty}\frac{1}{k}\mathbb{V}ar_t(x_{t+k} - x_t) = \lim_{k\to\infty}\left(1 + 2\sum_{j=1}^{k-1}\frac{(k-j)}{k}\rho_j\right)\sigma^2_{\Delta x} = S_{\Delta x}(e^{-i0}), \tag{9}$$

where ρ_j are the autocorrelations and $S_{\Delta x}(e^{-i\omega})$ is the spectral density function at frequency ω of (Δx_t). A further application of the Beveridge-Nelson decomposition implies

$$S_{\Delta x}(e^{-i0}) = \sigma^2_{\Delta z}. \tag{10}$$

Therefore, in order to estimate $\sigma^2_{\Delta z}$ we could use an estimator of the spectral density at frequency zero. However, estimating the spectral density function at low frequencies is extremely difficult and involves a trade-off between bias and efficiency of the estimator (see e.g. [15] §7.3 for such estimators and their properties). So, rather then relying on estimators for the spectral density function, we proceed directly with an estimator suggested by (8)-(10). In particular, (8) suggests to replace the autocovariance functions with their sample estimators and then employ well-known limit theorems for the sample autocovariances.

2.1 Large Sample Properties

In order to use (8) we recall that, under our assumptions, Δx is a covariance stationary process and, as such, has a moving average representation (1). Limit theorems for the sample autocovariances of such processes have been studied extensively (see [11], [14] §7, [15], §6) and we intend to utilize some of these results (much the same way as [18] did). Let us start by expressing the basic estimator

$$\bar{\sigma}^2_k = \frac{1}{Tk}\sum_{j=k}^{T}\left[(x_j - x_{j-k}) - \frac{k}{T}(x_T - x_0)\right]^2 \tag{11}$$

in a different form. Define $\hat{\epsilon}_j = x_j - x_{j-1} - \frac{1}{T}(x_T - x_0)$ then (11) becomes

$$
\begin{aligned}
\bar{\sigma}_k^2 &= \frac{1}{Tk}\sum_{j=k}^{T}\left[\sum_{l=1}^{k}(x_{j-k+l} - x_{j-k+l-1} - \frac{1}{T}(x_T - x_0))\right]^2 \\
&= \frac{1}{Tk}\sum_{j=k}^{T}\left[\sum_{l=1}^{k}\hat{\epsilon}_{j-k+l}\right]^2 \\
&= \frac{1}{Tk}\sum_{j=0}^{T-k}\left[\sum_{l=1}^{k}\hat{\epsilon}_{j+l}^2 + 2\sum_{l=1}^{k-1}\hat{\epsilon}_{j+l}\hat{\epsilon}_{j+l+1} + \ldots + 2\hat{\epsilon}_{j+1}\hat{\epsilon}_{j+k}\right] \\
&= \hat{\gamma}(0) + 2\frac{(k-1)}{k}\hat{\gamma}(1) + \ldots + \frac{2}{k}\hat{\gamma}(k-1) + \boldsymbol{o}(.)
\end{aligned}
$$

where

$$
\hat{\gamma}(h) = \frac{1}{T}\sum_{j=0}^{T-h}\hat{\epsilon}_j\hat{\epsilon}_{j+h}
$$

and $\boldsymbol{o}(.)$ specifies an error term in probability depending on the distribution of the innovations. Define the vector $\hat{\boldsymbol{\gamma}} = (\hat{\gamma}(0), \ldots, \hat{\gamma}(k-1))'$, then we can write

$$
\bar{\sigma}_k^2 = \boldsymbol{l}'\hat{\boldsymbol{\gamma}} + \boldsymbol{o}(.) \tag{12}
$$

with $\boldsymbol{l}$ the k-dimensional vector $\boldsymbol{l} = (1, 2\frac{(k-1)}{k}, \ldots, \frac{2}{k})'$. We therefore can use limit theorems on the asymptotic distribution of $\hat{\boldsymbol{\gamma}}$ to deduce the asymptotic distribution of our estimator $\bar{\sigma}_k^2$. These limit theorems depend crucially on the distribution of the innovations ϵ in (1). If $\mathbb{E}(\epsilon^4) < \infty$, the limit distribution of $\hat{\boldsymbol{\gamma}}$ is Gaussian, see e.g. [5], §7.3,§13.3, [15] §6.3. If $\mathbb{E}(\epsilon^4) = \infty, \sigma_\epsilon^2 < \infty$ (and further regularity conditions are satisfied), the limit distribution consists of a stable random variable multiplied by a constant vector, see [11] and [14] §7.3 for details. Hence, in the first case the asymptotic distribution of $\bar{\sigma}_k^2$ will be Gaussian, while in the second case it will asymptotically be distributed according to a stable law.

2.2 Small Sample Adjustments

In small samples, the estimator (11) exhibits a considerable bias. To discuss possible adjustments we assume that the data generating process is a pure unit root process, i.e. equation (5) becomes

$$\Delta x_t = \Delta z_t = \mu + \epsilon_t \tag{13}$$

with (ϵ_t) uncorrelated $(0, \sigma^2)$ random variables. So we can write the numerator of the estimator (11)

$$\begin{aligned} N_\sigma &= \sum_{j=k}^{T} \left((x_j - x_{j-k}) - \frac{k}{T}(x_T - x_0) \right)^2 \\ &= \sum_{j=k}^{T} \left(k\mu + \sum_{\nu=0}^{k-1} \epsilon_{j-\nu} - \frac{k}{T} \left(T\mu + \sum_{\nu=0}^{T-1} \epsilon_{T-\nu} \right) \right)^2 \\ &= \sum_{j=k}^{T} \left(\sum_{\nu=n-k+1}^{n} \epsilon_\nu - \frac{k}{T} \sum_{\nu=1}^{T} \epsilon_\nu \right)^2 . \end{aligned}$$

Defining $Z_{j,k} = \sum_{\nu=j-k+1}^{j} \epsilon_\nu$ and using the fact that the ϵ_ν are uncorrelated we get

$$\begin{aligned} \mathbb{E}(N_\sigma) &= \sum_{j=k}^{T} \left(\mathbb{E}(Z_{j,k}^2) - \frac{2k}{T} \mathbb{E}(Z_{j,k} Z_{T,T}) + \frac{k^2}{T^2} \mathbb{E}(Z_{T,T}^2) \right) \\ &= \mathbb{E}(\epsilon^2) \sum_{j=k}^{T} \left(k - \frac{2k^2}{T} + \frac{k^2}{T} \right) = \sigma^2 (T-k+1)(T-k)\frac{k}{T}. \end{aligned}$$

So in order to get an unbiased estimator for σ^2 using the quantity N_σ we have to multiply it by

$$\frac{T}{k(T-k+1)(T-k)},$$

which is just the adjustment proposed by Cochrane (compare [9]) and leads to

$$\hat{\sigma}_k^2 = \frac{T}{k(T-k)(T-k+1)} \sum_{j=k}^{T} \left[(x_j - x_{j-k}) - \frac{k}{T}(x_T - x_0) \right]^2 . \tag{14}$$

If we assume that the innovations in (13) are uncorrelated with existing fourth moment we can use the asymptotic equivalence of the estimators (11) and (14) to deduce the weak convergence[2]

$$\sqrt{T}(\hat{\sigma}_k^2 - \sigma^2) \Rightarrow N(0, \sigma^4((2k^2+1)/3k)). \tag{15}$$

[2] We denote weak convergence by "$\Rightarrow$".

If, however, the last existing moment of the innovations in (13) is of order $2 < \alpha < 4$, i.e the variance exists, but the fourth moment is infinite, we have the weak convergence

$$C(T,\alpha)\hat{\sigma}_k^2 \Rightarrow \sqrt{k}S, \tag{16}$$

where S is a stable random variable with index $\alpha/2$ and $C(T,\alpha)$ a constant depending on the T and the tail behaviour of the innovations, which is related to the index α. (The relevant asymptotic result for the autocovariances is Theorem 2.2 in [11], where the exact values of the constants to be used to construct the vector $\boldsymbol{l}$ in (12) can be found). If we drop the assumption (13) the limit laws remain of the same type. However, the variances change considerably since they depend on the autocovariances of the process.[3]

	$\hat{\sigma}_k^2$ [a]	**s.e.**	**s.e.**	$\hat{\sigma}_k^2$ [b]	**s.e.**	**s.e.**[c]	$\hat{\sigma}_k^2$ [d]	**s.e.**	**s.e.**[e]
w[f]	0.999	0.048	0.048	0.860	0.039	0.040	1.169	0.057	0.059
m	0.999	0.096	0.094	0.835	0.077	0.078	1.120	0.115	0.117
q	0.997	0.167	0.163	0.830	0.137	0.136	1.206	0.199	0.202
y	0.991	0.347	0.333	0.822	0.288	0.278	1.212	0.422	0.411

TABLE 1. Model with i.i.d. Gaussian innovations

[a]model: $\Delta x_t = \epsilon_t$ with $\epsilon_t \sim N(0,1)$ and $\sigma^2_{\Delta Z} = 1$. First s.e. column are always Monte-Carlo, second s.e. column are asymptotic s.e. assuming existence of the fourth moment.

[b]model: $\Delta x_t = a\Delta x_{t-1} + \epsilon_t$ with $\epsilon_t \sim N(0,1)$ and $\sigma^2_{\Delta z} = \left(\frac{1}{1-a}\right)^2 \sigma^2_\epsilon$, here $\sigma^2_{\Delta z} = 0.826$

[c]Adjusted for AR(1)-covariance structure

[d]model: $\Delta x_t = \epsilon_t + a\epsilon_{t-1}$ with $\epsilon_t \sim N(0,1)$ and $\sigma^2_{\Delta z} = (1-a)^2 \sigma^2_\epsilon$, here $\sigma^2_{\Delta z} = 1.21$

[e]Adjusted for MA(1)-covariance structure

[f]w=week, m=month, q=quarter, y=year

[3]For instance in case the $\mathbb{E}(\epsilon^4) = \eta\sigma^4$ we have $\lim_{n\to\infty} \boldsymbol{Cov}(\hat{\gamma(p)}\hat{\gamma(q)}) = (\eta - 3)\gamma(p)\gamma(q) + \sum_{k=-\infty}^{\infty}[(\gamma(k)\gamma(k-p+q) - \gamma(k+q)\gamma(k-p)]$ and thus $(\hat{\gamma_1},\ldots,\hat{\gamma_k})' \sim N((\hat{\gamma_1},\ldots,\hat{\gamma_k})', T^{-1}V)$, where V is the covariance matrix (see [5], §7.3). This implies an asymptotic standard normal distribution of $\hat{\sigma_k^2}$ with variance $\boldsymbol{l}'V\boldsymbol{l}$.

3 Monte Carlo Illustrations

In this section, we illustrate our estimating procedure using simulated time series. We consider three basic settings of first-difference stationary sequences with representation (1). First, as a benchmark case, we consider a pure random walk with representation as in (13). To study the effect of non-zero autocovariances of the series (Δx) on the asymptotic standard error, we simulate two further series, namely a sequence, whose first-difference follows an autoregressive model of order one (AR(1)-model) implying an infinite order moving average representation and on the other hand, a sequence, which has first-differences allowing a moving average representation of order one (MA(1)).

These settings imply that the error terms in (5) are perfectly correlated. The AR-model corresponds to a 'small' random walk component (in our setting it accounts for roughly 70% of variability of (x_k) in (5)). The MA-model, on the other hand, corresponds to a 'large' random walk component, the innovation variance of the random walk component (z_k) in (5) is larger (due to dependence) than the innovation variance of the series (x_k).

	$\hat{\sigma}_k^2$ [a]	**s.e.**	**s.e.**	$\hat{\sigma}_k^2$ [b]	**s.e.**	**s.e.**[c]	$\hat{\sigma}_k^2$ [d]	**s.e.**	**s.e.**[e]
w[f]	2.980	0.962	0.144	2.555	0.644	0.120	3.507	1.388	0.1779
m	2.977	0.983	0.283	2.483	0.667	0.237	3.602	1.458	0.349
q	2.970	1.023	0.490	2.467	0.753	0.490	3.618	1.467	0.605
y	2.992	1.406	1.000	2.464	1.107	0.834	3.621	1.868	1.234

TABLE 2. Model with i.i.d. $t(3)$ innovations

[a] model: $\Delta x_t = \epsilon_t$ with $\epsilon_t \sim t(3)$ and $\sigma^2_{\Delta Z} = 3$. First s.e. column are always Monte-Carlo, second s.e. column are asymptotic s.e. assuming existence of the fourth moment.

[b] model: $\Delta x_t = a\Delta x_{t-1} + \epsilon_t$ with $\epsilon_t \sim t(3)$ and $\sigma^2_{\Delta z} = \left(\frac{1}{1-a}\right)^2 \sigma^2_\epsilon$, here $\sigma^2_{\Delta z} = 2.479$

[c] Asymptotic standard error, adjusted for AR(1)-covariance structure

[d] model: $\Delta x_t = \epsilon_t + a\epsilon_{t-1}$ with $\epsilon_t \sim t(3)$ and $\sigma^2_{\Delta z} = (1-a)^2\,\sigma^2_\epsilon$, here $\sigma^2_{\Delta z} = 3.63$

[e] Asymptotic standard error, adjusted for MA(1)-covariance structure

[f] w=week, m=month, q=quarter, y=year

For each of these series, we consider three types of innovation process. As a standard model we consider i.i.d. Gaussian innovations.

Then we investigate the effect of heavy-tailed innovations using i.i.d. Student $t(3)$ innovations, and finally to discuss (second order) dependence we use GARCH(1,1)-innovations. Each experiment consisted of generating a series of length 3000 (with coefficients in line with coefficients obtained performing the corresponding ARIMA (-GARCH) for the series used in §4) and was repeated 5000 times. We report the mean of long-period volatility estimators for periods of length $k = 5, 20, 60, 250$ (weeks, month, quarters, years) together with standard errors (s.e.) computed from the Monte-Carlo simulations and according to the asymptotic results for an underlying pure unit root process with an existing fourth moment.

	$\hat{\sigma}_k^2$ [a]	**s.e.**	**s.e.**	$\hat{\sigma}_k^2$ [b]	**s.e.**	**s.e.**[c]	$\hat{\sigma}_k^2$ [d]	**s.e.**	**s.e.**[e]
w[f]	4.078	0.278	0.192	3.505	0.237	0.161	4.770	0.324	0.237
m	4.066	0.437	0.378	3.390	0.370	0.315	4.887	0.528	0.466
q	4.037	0.710	0.653	3.348	0.595	0.545	4.897	0.871	0.806
y	4.004	1.442	1.333	3.323	1.187	1.113	4.903	1.767	1.645

TABLE 3. Model with GARCH(1,1) innovations

[a] model: $\Delta x_t = \epsilon_t$ with $\epsilon_t \sim GARCH(1,1)$ and $\sigma^2_{\Delta Z} = 0.004$. First s.e. column are always Monte-Carlo, second s.e. column are asymptotic s.e. assuming existence of the fourth moment.

[b] model: $\Delta x_t = a\Delta x_{t-1} + \epsilon_t$ with $\epsilon_t \sim GARCH(1,1)$ and $\sigma^2_{\Delta z} = \left(\frac{1}{1-a}\right)^2 \sigma^2_\epsilon$, here $\sigma^2_{\Delta Z} = 0.003306$

[c] Adjusted for AR(1)-covariance structure

[d] model: $\Delta x_t = \epsilon_t + a\epsilon_{t-1}$ with $\epsilon_t \sim GARCH(1,1)$ and $\sigma^2_{\Delta z} = (1-a)^2 \sigma^2_\epsilon$, here $\sigma^2_{\Delta Z} = 0.0484$

[e] Adjusted for MA(1)-covariance structure

[f] w=week, m=month, q=quarter, y=year

In line with the asymptotic consistency of the estimator $\hat{\epsilon_k}^2$ (compare 8) the estimated value converges towards the true value of the innovation variance of the random walk component in all cases. For Gaussian and GARCH innovation (cases for which the appropriate limit theory holds) the asymptotic standard errors are in line with the observed Monte Carlo errors. As expected the asymptotic standard errors (calculated under the assumption of an existing fourth moment) become unreliable for heavy tailed innovation, i.e. simulations based on $t(3)$ innovations.

Since for shorter series the asymptotic standard error becomes unreliable we also tested various bootstrap based methods. Motivated by the application we have in mind we concentrated on series with length 1000 and standard normal or GARCH innovations. It turned out, that fitting a low-order AR-model to the simulated time series and resampling from the residuals produced satisfactory bootstrap standard errors.

Model	lag 60			lag 250		
	$\hat{\sigma}_k^2$ [a]	**B-s.e.**	**A-s.e.**	$\hat{\sigma}_k^2$	**B-s.e.**	**A-s.e.**
RW[b]	0.950	0.263	0.286	1.015	0.359	0.583
AR(1)	0.820	0.277	0.253	0.9314	0.668	0.823
MA(1)	1.199	0.349	0.363	1.270	0.816	0.841
RW[c]	3.886	1.163	1.117	3.997	2.634	2.366
AR(1)	3.282	0.960	0.952	3.041	1.887	1.926
MA(1)	4.702	1.311	1.395	4.814	2.823	2.946

TABLE 4. Bootstrap estimates of standard errors

[a] All on a time series of length 1000 with 5000 bootstrap resamples, parameters chosen as above

[b] standard Normal innovations

[c] GARCH-Innovations, values multiplied by 10^3

4 Applications

Empirical comparisons of continuous-time models of the short-term interest rate have recently been the focus of several studies (see e.g. [4, 6, 8, 10]). In these studies the general class of single-factor diffusion models

$$dr = (\mu - \kappa r)dt + \sigma r^{\gamma} dW, \tag{17}$$

with constant coefficients and W a standard Brownian motion has been compared. We will consider the subclass, where we restrict the parameter γ to take one of the values $0, 1/2$ or 1, so e.g. the Vasicek and the Cox-Ingersoll-Ross model are included. The discrete-time analog of this model class is

$$\begin{aligned} r_t - r_{t-1} &= \alpha + \beta r_{t-1} + \epsilon_t \\ \mathbb{E}(\epsilon_t | F_{t-1}) &= 0, \quad \mathbb{E}(\epsilon_t^2 | F_{t-1}) = \sigma^2 r_{t-1}^{2\gamma}, \end{aligned} \tag{18}$$

with F_t the information set at time t. A model like this will generate a time series within our framework if $\beta = 0$. If we focus on the unconditional long-term variance a standard calculation shows, that we have the following asymptotic relations (under $\beta = 0$)

$$\begin{array}{ll} \gamma = 0 & \mathbb{V}ar(r_t) \sim t \\ \gamma = \frac{1}{2} & \mathbb{V}ar(r_t) \sim t^2 \\ \gamma = 1 & \mathbb{V}ar(r_t) \sim e^{ct} \end{array}$$

(c a constant). Using the Cochrane-type estimator we can compare the observed long-term variances with variances predicted from the model setting. We apply this idea to three short-term (7 day-maturity) interest rate series. The rates we use are US EURO-DOLLAR (with 3512 observations from 01.01.85 – 18.06.98), UK EURO-POUND (with 3401 observations from 01.01.85 – 13.01.98), and German EURO-MARK (with 1222 observations from 09.01.95 – 14.09.99).

rate	$\hat{\sigma}_1^2$	$\hat{\sigma}_5^2$	$\hat{\sigma}_{20}^2$	$\hat{\sigma}_{60}^2$	$\hat{\sigma}_{250}^2$
US EURO-DOLLAR	0.0537	0.0438	0.0149	0.0077	0.0092
	(0.0055[a])	(0.0092)	(0.0107)	(0.0022)	(0.0051)
UK EURO-POUNDS$	0.0439	0.0293	0.0189	0.0169	0.0212
	(0.0051)	(0.0076)	(0.0123)	(0.0080)	(0.0118)
GER EURO-MARK$	0.0059	0.0048	0.0015	0.0013	0.0018
	(0.0031)	(0.0029)	(0.0009)	(0.0008)	(0.0008)

TABLE 5. Short rate volatilities

[a]For lags 1 to 20 s.e are based on asymptotic calculations, for lags 60 and 250 s.e. are bootstrap based

To ensure the validity of the assumption $\beta = 0$ we performed various tests for unit roots and stationarity[4]. For all series we can't

[4]For the unit root tests we used the augmented Dickey-Fuller and Phillips-Perron procedures and for testing stationarity the Kwiatkowski-Phillips-Schmidt-Sin test (see [19] chapters 3 and 4 for a description and discussion of these tests)

reject the presence of a unit root at a 10% significance level, whereas stationarity of the series is rejected at the 1% significance level. Applying these tests again to the first-difference of the series indicated no evidence of a unit root in the differenced series. The combination of these test results allows us to conclude the ball series should be modelled as first-difference stationary and fit into our framework.

We report the results for the interest series in table (5). From a model-free point of view (that is within the general framework (5)) these results indicate, that using the one-day volatility estimate will seriously overestimate longer term volatility.

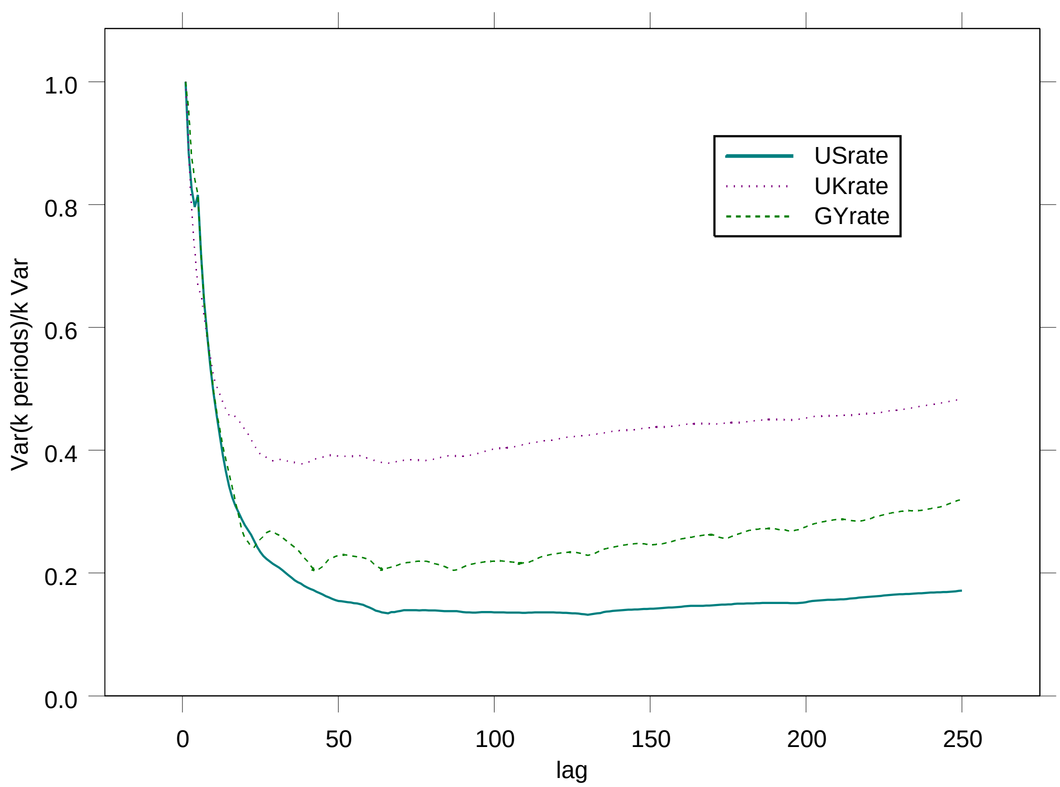

FIGURE 1. Variance-Ratios for short-term interest rates

Turning to the question of modelling short-term interest rates within the class of one-factor diffusion models we calculate and plot the ratio of the volatility calculated over a longer holding period to that calculated over one day multiplied by k (see figure 1). For all rates considered the ratios are downward slopping for short holding periods (the mean-reverting component dies off). After a period of stability the variance ratio begin to increase linearly showing a behaviour roughly in line with the asymptotics of a Cox-Ingersoll-Ross model.

5 Conclusion

We presented a non-parametric method to estimate long-term variances and the magnitude of the unit root process in various interest rates. Our results that calculating long-term variances on the basis of short-term variance estimates will overestimate long-term variances. Our results further indicate that within the one-factor diffusion short rate model class square-root type processes model the behaviour of long-term variances of short rates best. Vasicek-type models, which assume that the short rate follows a mean-reverting process and thus omit a unit root component in the data-generating process, will lead to an underestimating of long-term variances, since for longer time horizons the unit-root component of the interest-rate process becomes dominant. Our findings support a model of Cox-Ingersoll Ross type without a mean-reverting component.

6 References

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